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Entanglement swapping between multi-qudit systems

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We generalize the entanglement swapping scheme originally proposed for two pairs of qubits to an arbitrary number q of systems composed from an arbitrary number m_j of qudits. Each of the system is supposed to be prepared in a maximally entangled state of m_j qudits, while different systems are not correlated at all. We show that when a set $\sum_{j=1}^{q} a_j$ particles (from each of the q systems a_j particles are measured) is subjected to a generalized Bell-type measurement, the resulting set of $\sum_{j=1}^{q}(m_j - a_j)$ particles will collapse into a maximally entangled state.

I. INTRODUCTION

Recently quantum entanglement has been recognized as an important resource for quantum information processing. In particular, quantum computation [1, 2], quantum teleportation [3], quantum dense coding [4], certain types of quantum key distribution [5] and quantum secret sharing protocols [6] are rooted in the existence of quantum entanglement.

In spite of all the progress in the understanding of the nature of quantum entanglement there are still open questions which have to be answered. In particular, it is not clear yet how to uniquely quantify the degree of entanglement [7, 8, 9, 10, 11], or how to specify the inseparability conditions for bi-partite multi-level systems (qudits) [12]. A further problem which waits for a thorough illumination is the multiparticle entanglement [13]. There are several aspects of quantum multiparticle correlations, for instance the investigation of intrinsic n-party entanglement (i.e. generalizations of the GHZ state [14]). Another aspect of the multiparticle entanglement is that in contrast to classical correlation it cannot freely be shared among many objects [15, 16, 17, 18, 19].

In this paper we want to concentrate our attention on entanglement swapping. This is a method designed to entangle particles which have never interacted. The entanglement swapping has been proposed by Zukowski et al. [20] for two pairs of entangled qubits in one of the Bell states. Zeilinger et al. [21] have generalized the entanglement swapping to multiparticle systems. Bose et al. [22] proposed a different version of multiparticle entanglement swapping and suggested a few interesting ways of using this phenomenon. Bose et al. [23] investigated the purification protocol via entanglement swapping with non-maximally entangled states. This approach has been further improved by Shi et al. [24], and Hardy et al. [25]. Delayed choice entanglement swapping has been proposed and analyzed by Peres [26]. In [29, 30] the idea of entanglement swapping has been generalized to continuous variables. The use of entanglement swapping for purification in continuous dimension has been proposed by Parker et al. [27]. Entanglement swapping has been used not only for purification but also for cryptographic protocols (see, for instance, [28]). Finally, we note that entanglement swapping has been performed experimentally by Zeilinger et al. [31].

In this paper we will unify all theoretical approaches to the entanglement swapping in one generalized scheme. We present entanglement swapping for systems consisting of any number of entangled systems, each composed of an arbitrary number of qudits (i.e. quantum particles with Hilbert spaces of an arbitrary dimension D). This new unified approach allows us to discuss in detail various scenarios of multiparticle entanglement. Moreover, our formalism applies to all possible situations when quantum systems are maximally entangled. We do not discuss in this paper entanglement swapping between partially entangled systems.

In section II we present a relevant formalism for a description of kinematics of quantum states in D-dimensional Hilbert spaces. Section III serves as a simple introduction to our swapping scheme. We show how via a Bell-type measurement entanglement swapping can be realized. This idea is extended in section IV for the case of two entangled states, each having an arbitrary finite number of particles. In section V the most general entanglement scheme is presented. We summarize our results in section VI.

II. ENTANGLED QUDITS

Let the D-dimensional Hilbert space be spanned by D orthogonal normalized vectors $|x_k\rangle$, or, equivalently, by D vectors $|p_l\rangle$, $k,l = 0,\ldots,D-1$. These bases are related by the discrete Fourier transform

$$
\begin{aligned}
|x_k\rangle &= \frac{1}{\sqrt{D}}\sum_{l=0}^{D-1}\exp\Big(-i\frac{2\pi}{D}kl\Big)|p_l\rangle\,; \\
|p_l\rangle &= \frac{1}{\sqrt{D}}\sum_{k=0}^{D-1}\exp\Big(i\frac{2\pi}{D}kl\Big)|x_k\rangle\,. \qquad (1)
\end{aligned}
$$

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Without loss of generality, we assume that these bases are sets of eigenvectors of two non-commuting operators, the 'position' $\hat{x}$ and the 'momentum' $\hat{p}$, such that

$$\hat{x}|x_k\rangle = x_k|x_k\rangle\ , \quad \hat{p}|p_l\rangle = p_l|p_l\rangle\ , \tag{2}$$

where

$$x_k = L\sqrt{\frac{2\pi}{D}}k; \qquad p_l = \frac{\hbar}{L}\sqrt{\frac{2\pi}{D}}l\ . \tag{3}$$

The length, L can, for example, be taken to be equal to $\sqrt{\hbar/\omega m}$, where m is the mass and ω the frequency of a quantum 'harmonic' oscillator within a finite dimensional Fock space (in what follows we use units such that $\hbar = 1$).

Next we introduce operators which shift (cyclically permute) the basis vectors [32]:

$$\begin{aligned}\hat{R}_x(n)|x_k\rangle &= |x_{(k+n)\bmod D}\rangle\ ;\\ \hat{R}_p(m)|p_l\rangle &= |p_{(l+m)\bmod D}\rangle\ ,\end{aligned} \tag{4}$$

where the sums of indices are taken modulo D. In the x-basis these operators can be expressed as

$$\begin{aligned}\langle x_k|\hat{R}_x(n)|x_l\rangle &= \delta_{k+n,l}\ ;\\ \langle x_k|\hat{R}_p(m)|x_l\rangle &= \delta_{k,l}\exp\Big(i\frac{2\pi}{D}ml\Big)\ .\end{aligned} \tag{5}$$

Moreover these operators fulfil the Weyl commutation relation [33, 34, 35]

$$\hat{R}_x(n)\hat{R}_p(m) = \exp\Big(i\frac{2\pi}{D}mn\Big)\hat{R}_p(m)\hat{R}_x(n)\ , \tag{6}$$

although they do not commute; they form a representation of an Abelian group in a ray space. We can displace a state in arbitrary order using $\hat{R}_x(n)\hat{R}_p(m)$ or $\hat{R}_p(m)\hat{R}_x(n)$, the resulting state will be the same — the corresponding kets will differ only by an unimportant multiplicative factor. We see that the operators $\hat{R}_x(n)$ and $\hat{R}_p(m)$ displace states in the directions x and p, respectively. The product $\hat{R}_x(n)\hat{R}_p(m)$ acts as a displacement operator in the discrete phase space (k,l) [36]. These operators can be expressed via the generators of translations (shifts)

$$\begin{aligned}\hat{R}_x(n) &= \exp(-ix_n\hat{p})\ ;\\ \hat{R}_p(m) &= \exp(ip_m\hat{x})\ .\end{aligned} \tag{7}$$

We note that the structure of the group associated with the operators $\hat{R}_x(n)$ and $\hat{R}_p(m)$ is reminiscent of the group of phase-space translations (i.e., the Heisenberg group) in quantum mechanics [37].

Let us assume a system of two qudits each described by a vector in a D-dimensional Hilbert space $\mathcal{H}$. The tensor product of the two Hilbert spaces can be spanned by a set of D^2 maximally entangled two-qudit states (the analogue of the Bell basis for spin-$\frac{1}{2}$ particles) [37]

$$\begin{aligned}&|\psi(m,n)\rangle = \\ &\quad = \frac{1}{\sqrt{D}}\sum_{k=0}^{D-1}\exp\Big(i\frac{2\pi}{N}mk\Big)|x_k\rangle|x_{(k-n)\bmod N}\rangle\ ,\end{aligned} \tag{8}$$

where $m,n = 0,\ldots,D-1$. These states form an orthonormal basis in the space $\mathcal{H}\otimes\mathcal{H}$

$$\langle\psi(k,l)|\psi(m,n)\rangle = \delta_{k,m}\delta_{l,n}\ , \tag{9}$$

with

$$\sum_{m,n=0}^{D-1}|\psi(m,n)\rangle\langle\psi(m,n)| = \hat{I}\otimes\hat{I}\ . \tag{10}$$

In order to prove the above relations we have used the standard relation $\sum_{n=0}^{D-1}\exp[2\pi i(k-k')n/D] = D\delta_{k,k'}$.

It is interesting to note that the whole set of D^2 maximally entangled states $|\psi(m,n)\rangle$ can be generated from the state $|\psi(0,0)\rangle$ by the action of *local* unitary operations (shifts) of the form

$$|\psi(m,n)\rangle = \hat{R}_p(m)\otimes\hat{R}_x(n)|\psi(0,0)\rangle\ . \tag{11}$$

In what follows we shall simplify our notation. Because we will work mostly in the x-basis we shall use the notation $|x_k\rangle \equiv |k\rangle$. In addition we will use the notation $x\ominus y$ instead of $(x-y) \bmod D$. This serves to keep the derivations as synoptical as possible. Using this notation we can write down the maximally entangled state of two qudits as

$$|\psi(l,k)\rangle_{01} = \frac{1}{\sqrt{D}}\sum_{n=0}^{D-1}e^{\frac{i2\pi ln}{D}}|n\rangle_0|n\ominus k)\rangle_1, \tag{12}$$

where parameters k and l can take values between 0 and $D-1$.

In general, M-particle maximally entangled states can be written as

$$\begin{aligned}&|\Psi\rangle = |\psi(l,k_1,k_2,\ldots,k_{M-1})\rangle = \\ &\quad = \frac{1}{\sqrt{D}}\sum_{n=0}^{D-1}e^{\frac{i2\pi ln}{D}}|n\rangle_0\bigotimes_{i=1}^{M-1}|n\ominus k_i\rangle_i.\end{aligned} \tag{13}$$

These particles are entangled in the sense that tracing out any $(M-1)$ particles leaves the reduced density matrix of the remaining particle in a maximally mixed state described by the density operator $\frac{1}{D}I$.

III. TWO ENTANGLED PAIRS

First of all we study a simple example of entanglement swapping between two qutrits. Suppose we have two systems each composed of two entangled 3-dimensional pairs of particles. The two systems are not correlated at all and the state vector describing this composite system can be

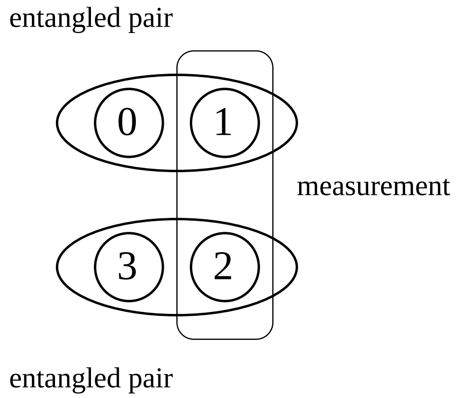

FIG. 1: A schematical description of entanglement swapping between two pairs of qubits (qudits). The qubits 1 and 2 are measured in the Bell basis. This measurement results in the entanglement of the qubits 0 and 3 which have never interacted directly.

expressed as

$$\begin{aligned}
|\Psi\rangle &= |\psi(0,0)\rangle_{01} \otimes |\psi(0,1)\rangle_{23} \qquad (14)\\
&= \frac{1}{\sqrt{3}}\left(|00\rangle_{01} + |11\rangle_{01} + |22\rangle_{01}\right)\\
&\quad \otimes \frac{1}{\sqrt{3}}\left(|02\rangle_{23} + |10\rangle_{23} + |21\rangle_{23}\right)\\
&= \frac{1}{3}\Big(|00\rangle_{01}|02\rangle_{23} + |00\rangle_{01}|10\rangle_{23} + |00\rangle_{01}|21\rangle_{23}\\
&\quad +|11\rangle_{01}|02\rangle_{23} + |11\rangle_{01}|10\rangle_{23} + |11\rangle_{01}|21\rangle_{23}\\
&\quad +|22\rangle_{01}|02\rangle_{23} + |22\rangle_{01}|10\rangle_{23} + |22\rangle_{01}|21\rangle_{23}\Big).
\end{aligned}$$

Now assume we perform a projective Bell-type measurement of particles 1 and 2 in the basis (12) with $D = 3$. If the measurement yields $|\psi(r,s)\rangle_{12}$ for some fixed r and s, the other two particles collapse into the state $|\psi(\tilde{l},\tilde{k})\rangle_{03}$. This result of the measurement conditionally 'selects' the vectors of the form

$$|n\rangle_0|n\rangle_1|n'\rangle_2|(n'-1) \bmod 3\rangle_3 \qquad (15)$$

for $n = 0\dots2$, such that $n' \equiv n - s \pmod 3$ and $\tilde{k} = s+1$. The amplitude of the vector $|n\rangle_1|(n-s) \bmod 3\rangle_2$ is $e^{i2\pi nr/3}$. It must hold that

$$e^{i2\pi n0/3}e^{i2\pi n'0/3} = e^0 = e^{i2\pi nr/3}e^{i2\pi n\tilde{l}/3}. \qquad (16)$$

Since $e^{ui2\pi} = e^{u'i2\pi}\,\forall u, u' \in \mathbb{Z}$ the equation (16) holds for $\tilde{l} = (-r) \bmod 3$. The previous derivations yield that the state of the particles 0 and 3 collapses into the maximally entangled state $|\psi((0-r) \bmod 3, (s+1) \bmod 3)\rangle_{03}$ of two qutrits.

Measuring a general state

Let us consider now a slightly more complex situation. We have a system of two entangled pairs in the general state $|\psi(l,k)\rangle_{01}\otimes|\psi(l',k')\rangle_{23}$. When we perform the measurement according to the basis (12) with $D = 3$ we obtain the vector $|\psi(r,s)\rangle_{12}$. The resulting state of particles 0 and 3 is again denoted as $|\psi(\tilde{l},\tilde{k})\rangle_{03}$. In this case we are looking for the vectors of the form

$$|n\rangle_0|n\ominus k\rangle_1|n'\rangle_2|n'\ominus k'\rangle_3 \qquad (17)$$

such that $n' \equiv n-k-s \pmod 3$, which yields $\tilde{k} = (k+s+k') \bmod 3$. The coefficient of the vector $|n\ominus k\rangle_1|n\ominus k\ominus s\rangle_2$ is $e^{i2\pi(n-k)r/3}$. It must hold as before (see equation (16)) that

$$e^{i2\pi nl/3}e^{i2\pi(n-k-s)l'/3} = e^{i2\pi(n-k)r/3}e^{i2\pi n\tilde{l}/3}e^{i2\pi x/3} \qquad (18)$$

for $n = 0,1,2$, where $e^{i2\pi x/3}$ will be part of the phase shift of the vector $|\psi(\tilde{l},\tilde{k})\rangle_{03}$. This implies the congruence

$$n(l+l'-r-\tilde{l}) \equiv -kr + kl' + sl' + x \pmod 3. \qquad (19)$$

The case $n = 0$ gives $-kr + kl' + sl' + x \equiv 0 \pmod 3$, so the x must be chosen such that this congruence is satisfied. For $n = 1$ this leads to a relation

$$\tilde{l} = (l + l' - r) \bmod 3. \qquad (20)$$

The extension to an arbitrary finite-dimensional systems is straightforward. It suffices to replace all ' mod 3' by ' mod D' and n varies from 0 to $D-1$. In equation (17) the generalization to D-dimensional system gives us $n' \equiv n - k - s \pmod D$. Since n varies from 0 to $D-1$, we have D vectors of the form (17). Therefore their linear combination with appropriate coefficients gives $|\psi(\tilde{l},\tilde{k})\rangle_{03}$ and not only a linear combination of less than D distinct vectors of the form $e^{i2\pi\tilde{l}n/D}|n\rangle_0|n\ominus\tilde{k}\rangle_3$. We can now summarize our results as follows.

Theorem 1 *Suppose that $|\Psi\rangle = |\psi(l,k)\rangle_{01}\otimes|\psi(l',k')\rangle_{23}$ is the tensor product of two maximally entangled pairs of qudits. Let assume that the particles 1 and 2 are measured via the Bell-type measurement in the basis (12). If the measurement yields the result $|\psi(r,s)\rangle_{12}$, then the two particles 0 and 3 collapse into the state*

$$|\psi((l+l'-r) \bmod D, (k+k'+s) \bmod D)\rangle_{03}. \qquad (21)$$

This is a maximally entangled state of qudits 0 and 3, which have never interacted before.

IV. ENTANGLING TWO MULTIPARTICLE SYSTEMS

Measurement of two particles

Suppose we have two uncorrelated systems of qudits. The first system with m_1+1 qudits is in a maximally entangled state $|\psi(l,k_1,\dots,k_{m_1})\rangle$, while the second system with m_2+1 qubits is in the state $|\psi(l',k'_1,\dots,k'_{m_2})\rangle$.

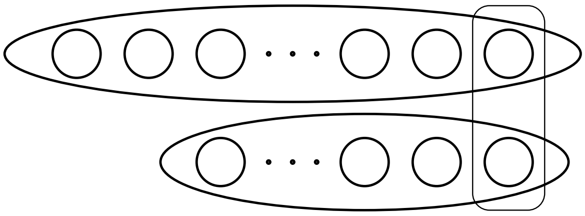

FIG. 2: A schematical description of entanglement swapping between two sets of entangled qudits. A single particle from each set is measured. This measurement results in entanglement between the rest of the particles from both of the systems.

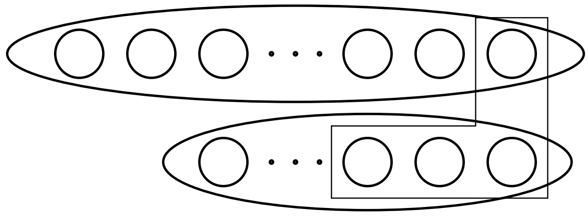

FIG. 3: The same as in figure 2 except an arbitrary number of particles from each set is measured.

The state vector of the composite system then reads $|\psi(l,k_1,\ldots,k_{m_1})\rangle \otimes |\psi(l',k'_1,\ldots,k'_{m_2})\rangle$. Now we can choose two arbitrary particles (one from each of the two systems) to be measured using the Bell-type projective measurement. Due to the cyclic symmetry we can assume that the 'last' particle of each of the two systems is measured. Suppose that in a measurement we obtain a state $|\psi(r,s)\rangle$. Therefore we are looking for vectors of the form

$$|n\rangle|n\ominus k_1\rangle\ldots|n\ominus k_{m_1}\rangle|n'\rangle\ldots|n'\ominus k'_{m_2}\rangle \tag{22}$$

such that $n'\equiv n-k_{m_1}-s+k'_{m_2} \pmod D$. To simplify the following derivations we put $k_0=k'_0=0$. Let

$$\Delta\tilde{k}=k_{m_1}+s-k'_{m_2}. \tag{23}$$

Now we should determine the $\tilde{l}$ and therefore

$$\begin{aligned} &e^{i2\pi nl/D}e^{i2\pi(n-k_{m_1}-s+k'_{m_2})l'/D}=\\ &\qquad = e^{i2\pi(n-k_{m_1})r/D}e^{i2\pi n\tilde{l}/D}e^{i2\pi x/D}. \end{aligned} \tag{24}$$

It follows that

$$\begin{aligned} &n(l+l'-r-\tilde{l})\equiv\\ &\quad\equiv -k_{m_1}r+k_{m_1}l'+sl'-k'_{m_2}l'+x \pmod D. \end{aligned} \tag{25}$$

As before (see equations (19) and (20)) for $n=1$ we have

$$\tilde{l}=(l+l'-r)\bmod D. \tag{26}$$

Once we have determined $\tilde{l}$, we can choose suitable x to satisfy the case $n=0$. This means the following congruence is equal to zero:

$$-k_{m_1}r+k_{m_1}l'+sl'-k'_{m_2}l'+x\equiv 0 \pmod D. \tag{27}$$

The resulting state is $|\psi(\tilde{l},\tilde{k}_1,\ldots,\tilde{k}_{m_1+m_2-1})\rangle$, where

$$\begin{aligned} \tilde{k}_i &= k_i && i<m_1\\ \tilde{k}_i &= k'_{i-m_1}+\Delta\tilde{k} && m_1\le i. \end{aligned}$$

In this section we have presented a technique which allows us to produce an entangled state with any number of particles.

Measuring more than two particles

Suppose that we are measuring the last a_1 particles of the first system and the last a_2 particles of the second system. We again assume the Bell-type measurement in the basis $|\psi(r,s_1,\ldots,s_{a_1+a_2-1})\rangle$ describing maximally entangled states of the a_1+a_2 qudits.

Analogically as in the previous examples we are looking for vectors of the form

$$\begin{aligned} &|n\rangle\ldots|n\ominus k_{m_1-a_1+1}\rangle\ldots|n\ominus k_{m_1}\rangle\otimes\\ &\qquad\otimes|n'\rangle\ldots|n'\ominus k'_{m_2-a_2+1}\rangle\ldots|n'\ominus k_{m_2}\rangle \end{aligned} \tag{28}$$

such that $n'\equiv n-k_{m_1-a_1+1}-s_{a_1}+k_{m_2-a_2+1}$ for a given result of the measurement $|\Psi\rangle=|\psi(r,s_1,\ldots,s_{a_1+a_2-1})\rangle$. Let $\Delta\tilde{k}=k_{m_1-a_1+1}+s_{a_1}-k_{m_2-a_2+1}$ and $k'_0=0$. Now let us determine $\tilde{l}$. It holds that

$$\begin{aligned} &e^{i2\pi(nl-x)/D}e^{i2\pi(n-k_{m_1-a_1+1}-s_{a_1}+k'_{m_2-a_2+1})l'/D}=\\ &\qquad = e^{i2\pi(n-k_{m_1-a_1+1})r/D}e^{i2\pi n\tilde{l}/D}. \end{aligned} \tag{29}$$

This leads again to the relation

$$\tilde{l}=(l+l'-r)\bmod D, \tag{30}$$

so the state of the unmeasured particles is $|\psi(\tilde{l},\tilde{k}_1,\ldots,\tilde{k}_{m_1+m_2+1-a_1-a_2})\rangle$, where

$$\begin{aligned} \tilde{k}_i &= k_i && i<m_1-a_1+1\\ \tilde{k}_i &= k'_{i-m_1+a_1-1}+\Delta\tilde{k} && m_1-a_1+1\le i. \end{aligned}$$

Theorem 2 *Suppose that we have two entangled systems with m_1+1 and m_2+1 particles, respectively, initially prepared in the state*

$$|\Psi\rangle=|\psi(l,k_1,\ldots,k_{m_1}\rangle\otimes|\psi(l',k'_1,\ldots,k'_{m_2}\rangle \tag{31}$$

and suppose that we subject the last a_1 particles from the first system and the last a_2 particles from the second system to a joint Bell-type measurement in the basis formed by vectors $|\psi(r,s_1,\ldots,s_{a_1+a_2-1})\rangle$. Then the vector describing the state of the remaining $m_1+m_2+2-a_1-a_2$ particles after the measurement is

$$|\psi(\tilde{l},\tilde{k}_1,\ldots,\tilde{k}_{m_1+m_2+1-a_1-a_2})\rangle \tag{32}$$

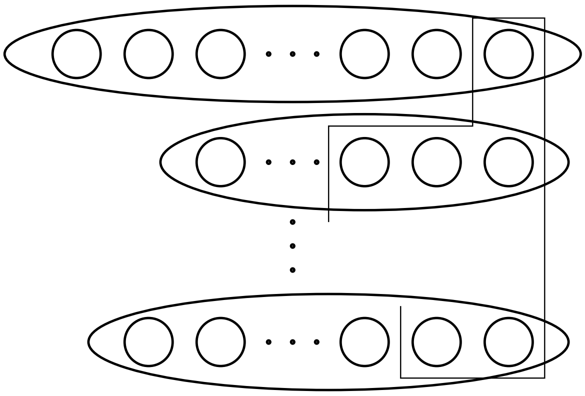

FIG. 4: The same as in figure 3 except that many initially uncorrelated multi-qudit systems are considered.

where

$$\begin{aligned} \tilde{k}_i &= k_i & i < m_1 - a_1 + 1 \\ \tilde{k}_i &= k'_{i-m_1+a_1-1} + \Delta\tilde{k} & m_1 - a_1 + 1 \leq i \\ \tilde{l} &= (l + l' - r) \bmod D. \end{aligned}$$

This means that the remaining particles end up in a maximally entangled state.

V. MANY MULTIPARTICLE ENTANGLED STATES

In what follows we describe the most general situation for entanglement swapping: Suppose we have q systems. The jth system is composed of $m_j + 1$ ($j = 1, \ldots, q$) particles which are in a maximally entangled state $|\psi(l^j, k_1^j, \ldots, k_{m_j}^j)\rangle$. The different systems are totally factorized, so the state vector of the composite system reads

$$|\Psi\rangle = \bigotimes_{j=1}^{q} |\psi(l^j, k_1^j, \ldots, k_{m_j}^j)\rangle. \tag{33}$$

(We note that superscripts do not denote the power, but they serve as indices.) Further we assume a multiparticle Bell-type measurement. Specifically, we consider a_j particles from jth state, $\forall j \in 1 \ldots q$, to be measured simultaneously in the basis

$$|\psi(r, s_1^1, \ldots, s_{a_1}^1, s_1^2, \ldots, s_{a_q-1}^q)\rangle. \tag{34}$$

The total number of measured particles is $\sum_{j=1}^q a_j$. After the measurement these particles collapse into one of the vectors (34). Therefore we look for the vectors

$$\begin{aligned} &|n^1\rangle|n^1 \ominus k_1^1\rangle \ldots |n^1 \ominus k_{m_1-a_1+1}^1\rangle \ldots |n^1 \ominus k_{m_1}^1\rangle \\ &\otimes \cdots \otimes |n^q\rangle \ldots |n^q \ominus k_{m_q-a_q+1}^q\rangle \ldots |n^q \ominus k_{m_q}^q\rangle \end{aligned} \tag{35}$$

such that

$$\begin{aligned} n^2 &\equiv n^1 - k_{m_1-a_1+1}^1 - s_{a_1}^1 + k_{m_2-a_2+1}^2 \pmod D \\ n^3 &\equiv n^1 - k_{m_1-a_1+1}^1 - s_{a_2}^2 + k_{m_3-a_3+1}^3 \pmod D \\ &\vdots \end{aligned} \tag{36}$$

which in general can be expressed as

$$n^i \equiv n^1 - k_{m_1-a_1+1} - s_{a_{i-1}}^{i-1} + k_{m_i-a_i+1}^i \pmod D \tag{37}$$

for $\forall i = 2 \ldots q$. It remains to determine $\tilde{l}$. As before we have

$$\prod_{j=1}^{q} e^{i2\pi n^j l^j/D} = e^{i2\pi(n^1 - k_{m_1-a_1+1})r/D} e^{i2\pi n^1 \tilde{l}/D} e^{i2\pi x/D}, \tag{38}$$

which yields

$$\begin{aligned} n^1 \left(\left(\sum_{j=1}^{q} l^j \right) - r - \tilde{l} \right) &\equiv -k_{m_1-a_1+1} r + x \\ + \sum_{j=2}^{q} l^j \left(k_{m_1-a_1+1}^1 + s_{a_{j-1}}^{j-1} - k_{m_j-a_j+1}^j \right) &\pmod D. \end{aligned} \tag{39}$$

The right-hand side of the congruence (39) is equal to 0 (mod D) which affects only the global phase. Therefore we can write

$$\tilde{l} = \left(\left(\sum_{j=1}^{q} l^j \right) - r \right) \mod D. \tag{40}$$

Consequently a set of $\sum_j (m_j - a_j + 1)$ unmeasured particles becomes entangled due to the Bell-type measurement performed on the $\sum_j a_j$ particles. The state of the unmeasured particles is

$$|\psi(\tilde{l}, \tilde{k}_1^1, \ldots, \tilde{k}_{m_1-a_1+1}^1, \tilde{k}_1^2, \ldots, \tilde{k}_{m_q-a_q}^q)\rangle. \tag{41}$$

Together there are $\left(\sum_{j=1}^q m_j - a_j + 1 \right) - 1$ $\tilde{k}$ and they must satisfy the condition

$$\begin{aligned} \tilde{k}_i^j &= k_i^j + n^j - n^1 & i \leq m_j - a_j \\ \tilde{k}_{m_j-a_j+1}^j &= n^{j+1} - n^1. \end{aligned} \tag{42}$$

Theorem 3 *Suppose we have q entangled systems each composed of $m_j + 1$ particles ($j = 1, \ldots, q$). Let the whole system is initially in the state* (33)*. Let us subject the last a_j particles from j-th ($\forall j = 1 \ldots q$) system to the Bell-type measurement in the basis formed by vectors* (34)*. Given the result of the measurement* (34) *the* $\sum_j (m_j -$

a_j+1) *unmeasured particles collapse into the maximally entangled state*

$$|\psi(\tilde{l},\tilde{k}_1^1,\ldots,\tilde{k}_{m_1-a_1+1}^1,\tilde{k}_1^2,\ldots,\tilde{k}_{m_q-a_q}^q)\rangle, \tag{43}$$

where

$$\tilde{l} = \left(\left(\sum_{j=1}^{q} l^j\right) - r\right) \mod D \tag{44}$$

and

$$\tilde{k}_i^j = k_i^j + n^j - n^1\,; \qquad i \le m_j - a_j;$$
$$\tilde{k}_{m_j-a_j+1}^j = n^{j+1} - n^1.$$

VI. CONCLUSION

In this paper we have presented a general formalism describing entanglement swapping between multi-qudit systems. We have shown that by performing Bell-type measurements one can create entangled states (with an arbitrary number of particles) from particles which have never interacted before.

Even though our formalism has been developed for finite-dimensional Hilbert space, it can be generalized for continuous variables, i.e. $D \to \infty$. In this case qudits are replaced by harmonic oscillators (e.g. quantized modes of an electromagnetic field). Formally, in the limit $D \to \infty$ we can substitute a two-qudit maximally entangled state by a two-mode correlated state, i.e.

$$\frac{1}{\sqrt{D}}\sum_n e^{ip_l x_n}|x_n\rangle|x_n - x_k\rangle \to |\psi(x,p)\rangle \tag{45}$$

where

$$|\psi(x,p)\rangle \equiv \frac{1}{\sqrt{2\pi}}\int d\tilde{x}e^{ip\tilde{x}}|\tilde{x}\rangle_0|\tilde{x}-x\rangle_1. \tag{46}$$

Analogously, a multi-mode entangled state in the continuous limit can be expressed as

$$\frac{1}{\sqrt{2\pi}}\int d\tilde{x}\; e^{ip\tilde{x}}|\tilde{x}\rangle_0 \bigotimes_{j=1}^{M-1}|\tilde{x}-x_j\rangle_j. \tag{47}$$

Once these states are defined one can formally perform the same manipulations as in the case of qudits, i.e. generalized Bell measurements, etc. Nevertheless, we remind ourselves that the maximally correlated states (45) as well as (47) require infinite energy for their creation. For this reason it is desirable to consider two-mode (and multi-mode) squeezed states which in the limit of infinite squeezing are equal to (45) and (47), respectively. It is convenient to describe these two mode state in term of their Wigner functions. In particular, the Wigner function corresponding to a regularized version of the state $|\psi(0,0)\rangle$ is [38]

$$W(x_1,p_1;x_2,p_2) = \exp\left\{-\frac{e^{2\xi}}{2}\left[(x_1-x_2)^2+(p_1+p_2)^2\right]\right\} \times \exp\left\{-\frac{e^{-2\xi}}{2}\left[(x_1+x_2)^2+(p_1-p_2)^2\right]\right\}. \tag{48}$$

This is a Wigner function describing a two-mode squeezed vacuum. If we trace over one of the modes, i.e., if we perform an integration over the parameters x_2 and p_2 we obtain from (48) a Wigner function of a thermal field where $\bar{n} = \sinh^2\xi$ is the mean excitation number in the two-mode squeezed vacuum under consideration. We note that the thermal state is a maximally mixed state (i.e. the state with the highest value of the von Neumann entropy) for a given mean excitation number. This means that the pure state (48) is the most entangled state for a given mean excitation number. From this it follows that to create a truly maximally entangled state, i.e. the state (48) in the limit $\xi \to \infty$, an infinite number of quanta is needed and so infinite energy.

From (48) one can easily find the Wigner functions of other states $|\psi(x,p)\rangle$. We remind ourselves that Wigner functions are invariant under canonical transformations (7). Taking into account that states $|\psi(x,p)\rangle$ can be obtained from $|\psi(0,0)\rangle$ by a canonical transformation (see (11))

$$|\psi(x,p)\rangle = \hat{R}_p(p)\otimes\hat{R}_x(x)|\psi(0,0)\rangle \tag{49}$$

its Wigner function can be obtained via a simple substitution of variables from the Wigner function (48). The generalized Bell measurement in this representation corresponds to a POVM measurement of the Artur-Kelly type [36]. This formalism in the infinite squeezing then leads to a perfect entanglement swapping between harmonic oscillators.

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